AUSLANDER CORRESPONDENCE FOR TRIANGULATED CATEGORIES

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Abstract. We give analogues of the Auslander correspondence for two classes of triangulated categories satisfying some finiteness conditions. We construct 'Auslander algebras' for each class and give homological characterizations of these Auslander algebras. We also present an application to Cohen-Macaulay representation theory.

1. Introduction

The classical Auslander correspondence, which can be regarded as 'for abelian categories', states a bijection between finite abelian categories and certain finite dimensional algebras. Let us first fix some notations to recall the precise statement.

Throughout this article, we denote by k a field. A 'category' means a k-linear, Hom-finite, Krull-Schmidt category. We say that a category is *finite* if it has only finitely many isomorphism classes of indecomposable objects. If $\mathcal{C}$ is a finite category, then it has an additive generator M, and we call its endomorphism algebra $\mathrm{End}_{\mathcal{C}}(M)$ the *Auslander algebra* of $\mathcal{C}$. This is uniquely determined by $\mathcal{C}$ up to Morita equivalence.

Now we can state the Auslander correspondence.

Theorem 1 ([2], Auslander correspondence)**.** *There exists a bijection between the following:*

1. *The set of equivalence classes of finite abelian categories $\mathcal{A}$.*
2. *The set of Morita equivalence classes of finite dimensional algebras Γ satisfying* $\mathrm{gl.dim}\,\Gamma \leq 2 \leq \mathrm{dom.dim}\,\Gamma$.

This theorem revealed the relationship between a categorical property (=being abelian) of $\mathcal{A}$ and homological invariants (=gl.dim and dom.dim) of Γ, which is now an important viewpoint in representation theory.

The aim of this note is to give a triangulated analogue of this theorem, along [4]. Namely, we give homological characterizations of triangulated categories satisfying certain finiteness conditions. We consider the following two kinds of finiteness on a triangulated category $\mathcal{T}$. The first one is that $\mathcal{T}$ is *finite*, which is a direct analogue of representation-finiteness. The second one is that $\mathcal{T}$ is '[1]-*finite*' which we introduce (see Definition 5 below). We treat the finite case in Section 2, [1]-finite case in Section 3, and present an application in the final section.

The detailed version of this paper has been submitted for publication elsewhere.

2. Finite case

In this section, we give an Auslander correspondence for finite triangulated categories and a sketch of its proof. We have the following homological characterization of the Auslander algebras of finite triangulated categories.

Theorem 2. *Let k be perfect field. Then, the following are equivalent for a basic finite dimensional k-algebra A:*

(1) *A is the basic Auslander algebra of a finite triangulated category.*
(2) *A is self-injective and $\Omega^3 \simeq (-)_\alpha$ on $\underline{\mathrm{mod}}\, A$ for some automorphism α of A.*

The implication (1) $\Rightarrow$ (2) is rather easy, so we give a proof here.

Proof of (1) $\Rightarrow$ (2). Let $\mathcal{T}$ be a finite triangulated category with an additive generator M. Take M to be basic and set $A = \mathrm{End}_{\mathcal{T}}(M)$ so that A is the basic Auslander algebra of $\mathcal{T}$. The self-injectivity of A is a consequence of the well-known fact that a triangulated category is self-injective. Also each triangle $X \to Y \to Z \to$ in $\mathcal{T}$ yields an exact sequence

$$\begin{tikzcd}[column sep=small] (-,Z[-1]) \arrow[rr, tail] \arrow[dr] & & (-,X) \arrow[r] & (-,Y) \arrow[r] & (-,Z) \arrow[rr, two heads] \arrow[dr] & & (-,X[1]) \\ & \Omega^3 L \arrow[ur] & & & & L \arrow[ur] & \end{tikzcd}$$

in $\mathrm{mod}\,\mathcal{T} = \mathrm{mod}\, A$, where we write $(-,-)$ for $\mathrm{Hom}_{\mathcal{T}}(-,-)$. Therefore the third syzygy on $\underline{\mathrm{mod}}\, A$ is induced by $[-1]$ on $\mathcal{T}$. Since M is basic, $[1]$ induces an automorphism α of A such that $\Omega^3 \simeq (-)_\alpha$. $\square$

For the converse implication, we need a result which allows one to introduce a triangle structure on certain additive categories. We can use the following result due to Amiot.

Proposition 3 ([1])**.** *Let $\mathcal{A}$ be a k-linear category such that* $\mathrm{mod}\,\mathcal{A}$ *is naturally Frobenius. Let S be an automorphism of $\mathcal{A}$ and extend this to* $\mathrm{mod}\,\mathcal{A} \to \mathrm{mod}\,\mathcal{A}$ *(by $M \mapsto M \circ S^{-1}$). Assume there exists an exact sequence*

$$0 \longrightarrow 1 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0$$

of exact functors from $\mathrm{mod}\,\mathcal{A}$ *to* $\mathrm{mod}\,\mathcal{A}$ *such that the X^i's values in $\mathcal{A}$. Then, $\mathcal{A}$ has a structure of a triangulated category with suspension S. In this case, the triangles are given by $X^0M \to X^1M \to X^2M \to SX^0M$ with $M \in \mathrm{mod}\,\mathcal{A}$.*

Such an exact sequence of functors can be obtained by considering bimodules as functors. We have the following variation of the result of Green-Snashall-Solberg, which relates our homological condition (2) in Theorem 2 of 'Auslander algebras' with bimodules. We denote by J_A the Jacobson radical of an algebra A, and by $A^e = A^{\mathrm{op}} \otimes_k A$ the enveloping algebra.

Proposition 4 (a variant of [3])**.** *Let A be a ring-indecomposable non-semisimple finite dimensional k-algebra such that A/J_A is separable over k and $n > 0$. Then, the following are equivalent.*

(1) *A is self-injective and $\Omega^n \simeq (-)_\alpha$ on $\underline{\mathrm{mod}}\, A$ for some automorphism α of A.*
(2) *There exists an automorphism of α of A such that $\Omega^n_{A^e}(A) \simeq {}_1A_\alpha$ in* $\mathrm{mod}\, A^e$.

Now the implication (2) $\Rightarrow$ (1) is a consequence of Propositions 3 and 4.

3. [1]-finite case

Our next finiteness condition on triangulated categories is [1]-finiteness. We first introduce the notion and note a consequence.

Definition 5. A triangulated category $\mathcal{T}$ is [1]-*finite* if it satisfies the following conditions:

(1) There exists $M \in \mathcal{T}$ such that $\mathcal{T} = \mathrm{add}\{M[i] \mid i \in \mathbb{Z}\}$.

(2) For any $X, Y \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for almost all $i \in \mathbb{Z}$.

In this case, we say M as in (1) a [1]-*additive generator* for $\mathcal{T}$.

For example, the bounded derived category $D^b(\mathrm{mod}\, kQ)$ of the path algebra of a Dynkin quiver Q is [1]-finite, and an additive generator for $\mathrm{mod}\, kQ$ is a [1]-additive generator for $D^b(\mathrm{mod}\, kQ)$.

The result due to Xiao-Zhu and Riedtmann's 'knitting' argument gives a classification of [1]-finite triangulated categories, as *additive* categories.

Proposition 6 ([5, 6])**.** *Let k be an algebraically closed field and $\mathcal{T}$ be a* [1]*-finite triangulated category over k.*

(1) *The AR-quiver of $\mathcal{T}$ is $\mathbb{Z}Q$ for some Dynkin diagram Q.*

(2) *$\mathcal{T}$ is standard, that is, $\mathcal{T}$ is k-linearly equivalent to the mesh category $k(\mathbb{Z}Q)$.*

Next we discuss what the 'Auslander algebras' of [1]-finite triangulated categories should be. The construction is based on the following graded version of so-called 'projectivization'.

Proposition 7. *Let $\mathcal{C}$ be a category with an automorphism F. Assume $\mathcal{C} = \mathrm{add}\{F^iM \mid i \in \mathbb{Z}\}$ for some $M \in \mathcal{C}$. Set $\Gamma = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, F^iM)$. Then, there exists an equivalence*

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, F^i(-))\colon\ \mathcal{C} \xrightarrow{\ \simeq\ } \mathrm{proj}^{\mathbb{Z}}\Gamma \qquad \mathcal{C} \circlearrowright F, \quad \mathrm{proj}^{\mathbb{Z}}\Gamma \circlearrowright (1)$$

such that the action of F on $\mathcal{C}$ is compatible with the degree shift (1) *on* $\mathrm{proj}^{\mathbb{Z}}\Gamma$*.*

Γ does not depend on the choice of M up to graded Morita equivalence in the sense that if Γ' is obtained from another choice M' of M, then there is an equivalence $\mathrm{Mod}^{\mathbb{Z}}\Gamma \simeq \mathrm{Mod}^{\mathbb{Z}}\Gamma'$ compatible with each degree shift.

Applying this construction to [1]-finite triangulated categories, we have the following notion of [1]-*Auslander algebras*.

Definition 8. Let $\mathcal{T}$ be a [1]-finite triangulated category with a [1]-additive generator M. We call

$$C = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(M, M[i])$$

the [1]-*Auslander algebra* of $\mathcal{T}$.

Note that C is finite dimensional thanks to the condition (2) in Definition 5. Also, as in the proof of Theorem 2 (1) $\Rightarrow$ (2), we see that C is self-injective and satisfies $\Omega^3 \simeq (-1)$ on $\underline{\mathrm{mod}}^{\mathbb{Z}}C$.

We now have the Auslander correspondence for [1]-finite triangulated categories.

Theorem 9. *Let k be an algebraically closed field. There exists a bijection between the following.*

(1) *The set of triangle equivalence classes of k-linear,* Hom*-finite algebraic triangulated categories $\mathcal{T}$ which are* $[1]$*-finite.*
(2) *The set of graded Morita equivalence classes of finite dimensional graded self-injective algebras C such that $\Omega^3 \simeq (-1)$ on $\underline{\mathrm{mod}}^{\mathbb{Z}} C$.*
(3) *The set of disjoint unions on Dynkin diagrams of type A, D and E.*

The correspondences are given as follows:

- *From (1) to (2): taking the* $[1]$*-Auslander algebra.*
- *From (2) to (1): $C \mapsto \mathrm{proj}^{\mathbb{Z}} C$.*
- *From (1) to (3): taking the tree type of the AR-quiver of $\mathcal{T}$.*
- *From (3) to (1): $Q \mapsto k(\mathbb{Z}Q)$.*

Note that this Theorem gives a classification of algebraic $[1]$-finite triangulated categories as *triangulated* categories by Dynkin diagrams. (Compare with Proposition 6.) This is a consequence of a result which states the uniqueness of triangle structures on mesh categories, or more generally, the homotopy category $K^b(\mathrm{proj}\, R)$ for certain rings R, see [4] for details. We end this section by stating a classification theorem of $[1]$-finite algebraic triangulated categories.

Theorem 10. *Any algebraic* $[1]$*-finite triangulated category over an algebraically closed field k is triangle equivalent to a the derived category $D^b(\mathrm{mod}\, kQ)$ of the path algebra kQ for some Dynkin quiver Q.*

4. Application to Cohen-Macaulay modules

In this final section, we give an application of the classification Theorem 10 to Cohen-Macaulay representation theory. Let Λ be an Iwanaga-Gorenstein algebra, that is, a Noetherian algebra whose self-injective dimension is finite on each side. Such algebras have a nice class of modules, called the *Cohen-Macaulay* modules, defined as follows:

$$\mathrm{CM}\, \Lambda = \{X \in \mathrm{mod}\, \Lambda \mid \mathrm{Ext}^i_\Lambda(X, \Lambda) = 0 \text{ for all } i > 0\}.$$

This is naturally a Frobenius category, hence its stable category $\underline{\mathrm{CM}}\, \Lambda$ is triangulated. If Λ is a graded Iwanaga-Gorenstein algebra, we similarly have the category

$$\mathrm{CM}^{\mathbb{Z}} \Lambda = \{X \in \mathrm{mod}^{\mathbb{Z}} \Lambda \mid \mathrm{Ext}^i_\Lambda(X, \Lambda) = 0 \text{ for all } i > 0\}$$

of graded Cohen-Macaulay modules, which again yields a triangulated category $\underline{\mathrm{CM}}^{\mathbb{Z}} \Lambda$.

For our result, we impose some assumptions on an Iwanaga-Gorenstein algebra Λ:

Assumption 11. (1) $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ is positively graded and each Λ_i is finite dimensional over k.
(2) The degree 0 part Λ_0 has finite global dimension.

We say that a graded Iwanaga-Gorenstein algebra Λ is *CM-finite* if $\mathrm{CM}^{\mathbb{Z}} \Lambda$ has only finitely many indecomposable objects up to isomorphism and degree shift.

We now have the following result.

Theorem 12. *Let Λ be a graded Iwanaga-Gorenstein algebra with the above assumptions. Suppose that Λ is CM-finite. Then, the triangulated category $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda$ is $[1]$-finite. Therefore, if k is algebraically closed, there exists a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda \simeq D^b(\mathrm{mod}\, kQ)$ for some Dynkin quiver Q.*

This theorem applies in particular to, for example,

- (commutative) simple singularities,
- representation-finite self-injective algebras,
- CM-finite Gorenstein orders.

References

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